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Abelian functions for cyclic trigonal curves of genus 4

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Abstract

We discuss the theory of generalized Weierstrass σ and $\wp$-functions defined on a trigonal curve of genus 4, following earlier work on the genus 3 case. The specific example of the "purely trigonal" (or "cyclic trigonal") curve $y^3 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ is discussed in detail, including a list of some of the associated partial differential equations satisfied by the $\wp$-functions, and the derivation of addition formulae.

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1. Introduction

Over the last few years there has been increasing interest in explicit descriptions of Abelian functions of algebraic curves. The beginnings of this theory go back to Weierstrass's theory of elliptic functions, and we will take this as our model. The key results of this are simply stated: let $\sigma(u)$ and $\wp(u)$ be the standard functions in Weierstrass elliptic function theory. They satisfy the well-known formulae

$$\wp(u) = -\frac{\mathrm{d}^2}{\mathrm{d}u^2} \ln \sigma(u), \tag{1.1}$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{1.2}$$

$$\wp'' = 6\wp^2 - \frac{1}{2}g_2. \tag{1.3}$$

The σ-function has a power series expansion, originally due to Weierstrass:

$$\sigma(z) = u - \frac{1}{240}g_2u^5 - \frac{1}{840}g_3u^7 - \frac{1}{161280}{g_2}^2u^9 - \frac{1}{2217600}g_2g_3u^{11} + \cdots \tag{1.4}$$

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Here the coefficients satisfy a linear recurrence, which is given in [1] (see also [12]).

The σ-function for an elliptic curve also satisfies a two-term addition formula which is a key result of the theory:

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v). \tag{1.5}$$

Taking logarithmic derivatives, one may obtain the standard algebraic addition formula for $\wp(u)$.

Klein's generalization of this theory is described in the classical monograph of Baker [3]; in particular, in [5] he worked out in detail the formulae corresponding to these for a genus 2 hyperelliptic curve. More recently such addition formulae for other curves have been derived; for example [6], where it was shown that for a hyperelliptic curve of arbitrary genus, the right hand side of the two-term addition formula for σ generalizes to an expression written in terms of a Pfaffian.

The corresponding theory for trigonal curves is more complex. Few examples have been worked out explicitly: the σ-function realization of the Abelian functions of a trigonal curve was developed in some detail in [8,13] and some of the present authors [9,14,18], have studied specific examples. These examples have concentrated on cyclic trigonal curves. This subset of curves possesses symmetry properties which simplify the calculations considerably, thus providing a useful starting point to investigate the properties of Abelian functions of trigonal curves in more detail.

There is a natural splitting of $(3, s)$ cyclic trigonal curves, with $s \neq 0 \bmod 3$, into those with $s = 3n + 1$ and those with $s = 3n + 2$. The simplest non-trivial case of this first class, $s = 3n + 1$, is the $(3, 4)$ curve of genus 3. The two papers [14,18] considered the general case for this curve and then looked in greater detail at the corresponding cyclic curve. For the cyclic curve, they found not only an explicit two-term addition formula, which has the same left hand side as that of the Weierstrass' addition formula (1.5), but also a three-term addition formula. This three-term addition formula reflects the natural symmetries of the cyclic curve. It gives an expansion of

$$\frac{\sigma(u+v)\sigma(u+[\zeta]v)\sigma(u+[\zeta^2]v)}{\sigma(u)^3\sigma(v)^3}, \quad \zeta = \exp\left(\frac{2\pi i}{3}\right)$$

in terms of Abelian functions of u and v. A key tool in calculating these addition formulae was the evaluation of two bases of linearly independent Abelian functions. The first of these bases has poles of order at most two on the Θ-divisor, where σ has a simple zero, and the second basis has poles of order at most 3 on the Θ-divisor.

The next case to consider is the simplest non-trivial curve of the second class: a $(3, 5)$ trigonal curve with genus 4. We should stress here that the $(3, 4)$ and $(3, 5)$ cases have some important differences. Most immediate is the fact that for the $(3, 4)$ curve, σ is an odd function of u, while for the $(3, 5)$ case it is even. More generally for a $(3, s)$ curve where s is not divisible by 3, σ has odd or even parity for s respectively even or odd. This change of parity has consequences in subsequent formulae, most clearly in the two-term addition formulae — the right hand side is respectively antisymmetric or symmetric between u and v in the $(3, 4)$ and $(3, 5)$ cases.

The cyclic $(3, 5)$ curve was considered in [9] and is the topic of this paper. A series expansion of the σ-function for the $(3, 5)$ curve was found by two of the present authors in [9], and that series, extended to higher order, plays a crucial role in some of the proofs below. They also found explicit formulae for a basis of differentials and for the Jacobi inversion formula for the curve. A basis of Abelian functions with second-order poles has been constructed, leading to sets of differential equations satisfied by the fundamental Abelian functions of the curve; these are of three kinds, those expressible as a fourth-order quasilinear PDE for σ, and those expressible as third-order PDE for σ, either quasilinear or else of second degree.

These results have also enabled us to find an explicit two-term addition theorem. Further, it has been possible to demonstrate that a three-term addition theorem exists for this curve and indeed for a cyclic trigonal curve of any genus. However, in contrast to the $(3, 4)$ case, a basis of Abelian functions with third-order poles has not been explicitly constructed, so no explicit form for the three-term addition formula can yet be found.

As commented in [14], our study is far from complete. One problem still to be considered is an explicit recursive construction of the σ-series generalizing the one given by Weierstrass. For hyperelliptic curves of genus 2 this was done in [10]. Another goal is to understand the algebraic structure of the two kinds of addition theorem developed here, in order to generalize results to higher genera, as [6] did for hyperelliptic curves. Buchstaber and Leykin have described some progress on these problems for general curves in [11].

Section 2 of this paper reviews the relevant theory of holomorphic differentials on a trigonal, and particularly a cyclic trigonal curve; in Section 3 the σ-function is introduced and its properties discussed — following Klein, our discussion uses the fundamental bidifferential, which plays a major role in what follows. In Section 4 we introduce the fundamental Abelian functions $\wp_{ij}$ on the curve as well as some important differential expressions in these, denoted as $Q_{ijk\ell}$. We then expand the Klein bidifferential in two ways to obtain the Jacobi inversion formula and some simple examples of the two kinds of PDE satisfied by the $\wp_{ij}$. Section 5 deals with the vector space $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$, of functions with at most double poles on the Θ-divisor, where $\sigma(u)$ vanishes. We construct an explicit basis for this space. In Section 6 we describe the Taylor expansion of $\sigma(u)$ near the origin, first given in [9]. This is then used, with the results of Section 5, to obtain the PDEs satisfied by the Abelian functions, of which the known examples are listed in Section 7. Finally Sections 8 and 9 are concerned respectively with the two- and three-term addition formulae satisfied by $\sigma(u)$ — the results of Section 5 enable the two-term formula in Section 8 to be given explicitly.

2. Trigonal curves of degree five

We define a general trigonal curve with the unique branch point ∞ at infinity by $g(x, y) = 0$, where

$$\begin{aligned} g(x, y) &= y^3 + (\mu_2 x + \mu_5)y^2 + (\mu_1 x^3 + \mu_4 x^2 + \mu_7 x + \mu_{10})y \\ &\quad - (x^5 + \mu_3 x^4 + \mu_6 x^3 + \mu_9 x^2 + \mu_{12} x + \mu_{15}) \quad (\mu_j \text{ are constants}). \end{aligned} \tag{2.1}$$

This curve is of genus 4, if it is non-singular. The topic of this paper is a special subclass of this family, the curves C of the form $f(x, y) = 0$, where

$$f(x, y) = y^3 - (x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0), \quad (\lambda_j \text{ are constants}). \tag{2.2}$$

A curve of this form is called a *cyclic* trigonal curve, [2]; such curves are invariant under the cyclic symmetry

$$[\zeta] : (x, y) \to (x, \zeta y),$$

where ζ is a cube root of unity. In [18,14] these curves are called *purely* trigonal, to distinguish them from other trigonal curves invariant under a cyclic group. This symmetry plays a significant role in what follows. All objects of the theory must transform simply under this group action. The argument we use here is closely modelled on that of [14] which studied the analogous problem for a cyclic (3, 4) curve.

We consider the set of differentials $\omega_1, \ldots, \omega_4$ where

$$\omega_1 = \frac{\mathrm{d}x}{\frac{\partial}{\partial y} f(x, y)}, \qquad \omega_2 = \frac{x\mathrm{d}x}{\frac{\partial}{\partial y} f(x, y)}, \qquad \omega_3 = \frac{y\mathrm{d}x}{\frac{\partial}{\partial y} f(x, y)}, \qquad \omega_4 = \frac{x^2\mathrm{d}x}{\frac{\partial}{\partial y} f(x, y)}. \tag{2.3}$$

This is a basis of the space of differentials of the first kind on C. We denote the vector consisting of the forms (2.3) by

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \tag{2.4}$$

From the general theory, we know that for four variable points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) on C, the sum of integrals from ∞ to these four points

$$\begin{aligned} u &= (u_1, u_2, u_3, u_4) \\ &= \int_\infty^{(x_1,y_1)} \omega + \int_\infty^{(x_2,y_2)} \omega + \int_\infty^{(x_3,y_3)} \omega + \int_\infty^{(x_4,y_4)} \omega \end{aligned} \tag{2.5}$$

fills the whole space $\mathbf{C}^4$. We denote the points in $\mathbf{C}^4$ by, for example, u and v and their natural coordinates in $\mathbf{C}^4$ by the subscripts (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4). We denote the lattice generated by the integrals of the basis (2.3) along any closed paths on C by Λ. We denote the manifold $\mathbf{C}^4/\Lambda$, by J, the Jacobian variety of C. The projection from $\mathbf{C}^4$ to $\mathbf{C}^4/\Lambda$ is denoted by κ:

$$\kappa : \mathbf{C}^4 \to \mathbf{C}^4/\Lambda = J. \tag{2.6}$$

We have $\Lambda = \kappa^{-1}((0, 0, 0, 0))$. We define for $k = 1, 2, 3, \ldots,$ the Abel map

$$\begin{aligned} \iota &: \mathrm{Sym}^k(C) \to J \\ (P_1, \cdots, P_k) &\mapsto \left(\int_\infty^{P_1} \omega + \cdots + \int_\infty^{P_k} \omega \right) \bmod \Lambda, \end{aligned} \tag{2.7}$$

and denote its image by $W^{[k]}$. Let

$$[-1](u_1, u_2, u_3, u_4) = (-u_1, -u_2, -u_3, -u_4), \tag{2.8}$$

and

$$\Theta^{[k]} := W^{[k]} \cup [-1]W^{[k]}. \tag{2.9}$$

We call this $\Theta^{[k]}$ the k-th *standard theta subset*. In particular, if $k = 1$, then (2.7) gives an embedding of C:

$$\begin{aligned} \iota &: C \to J \\ P &\mapsto \int_\infty^{P} \omega \bmod \Lambda. \end{aligned} \tag{2.10}$$

We note that in contrast to the hyperelliptic case

$$\Theta^{[2]} \neq W^{[2]}, \qquad \Theta^{[1]} \neq W^{[1]}. \tag{2.11}$$

On the embedded surface $\iota(C) = W^{[1]}$, we can take u_4 as a local parameter at the origin $\iota(\infty)$. Then we have (see [9, 13], for instance) Laurent expansions with respect to u_4 as follows:

$$u_1 = \frac{1}{7}{u_4}^7 + \cdots, \quad u_2 = \frac{1}{4}{u_4}^4 + \cdots, \qquad u_3 = \frac{1}{2}{u_4}^2 + \cdots \tag{2.12}$$

and also

$$x(u) = \frac{1}{{u_4}^3} + \cdots, \qquad y(u) = \frac{1}{{u_4}^5} + \cdots. \tag{2.13}$$

As with the (3, 4) curve, we introduce a set of weights for the different variables, as follows:

Definition 2.1. We define a weight called the *Sato weight* for constants and variables appearing in our relations as follows. The Sato weights of variables u_1, u_2, u_3, u_4 are 7, 4, 2, 1, respectively; the Sato weight of each coefficient λ_j in (2.2) is $15 - 3j$; while the Sato weights of $x(u)$ and $y(u)$ are -5 and -3, respectively.

We note that the Sato weights of the variables u_k are precisely the Weierstrass gap numbers of the Weierstrass gap sequence at ∞, whilst the Sato weights of $x(u)$ and $y(u)$ are Weierstrass non-gap numbers from the same sequence.

All expressions in this paper are homogeneous with respect to this weight.

3. The sigma function

We now construct the *sigma function*

$$\sigma(u) = \sigma(u_1, u_2, u_3, u_4) \tag{3.1}$$

associated with C (see also [6], Chap.1). We choose a basis of cycles on C:

$$\alpha_i, \beta_j \quad (1 \leqq i, j \leqq 4), \tag{3.2}$$

such that their intersection numbers are

$$\begin{aligned} \alpha_i \cdot \alpha_j &= \beta_i \cdot \beta_j = 0, \\ \alpha_i \cdot \beta_j &= \delta_{ij}. \end{aligned}$$

Let Z and W be two indeterminates. We define

$$\Omega\left((x,y),(z,w)\right)=\frac{1}{(x-z)\frac{\partial}{\partial y}f(x,y)}\sum_{k=1}^{3}y^{3-k}\left[\frac{f(Z,W)}{W^{3-k+1}}\right]_W\Bigg|_{(Z,W)=(z,w)}, \tag{3.3}$$

where $[\,]_W$ means removing the terms of negative powers with respect to W.

Lemma 3.1. *The fundamental* 2*-form of the second kind.*
Let

$$((x,y),(z,w))\mapsto R\left((z,w),(x,y)\right)\mathrm{d}z\mathrm{d}x \tag{3.4}$$

be a 2*-form on* $C\times C$ *with only poles along the diagonal points* $\{((x,y),(x,y))\}\subset C\times C$, *holomorphic elsewhere and satisfying*

$$\lim_{x\to z}(x-z)^2R\left((z,w),(x,y)\right)=1. \tag{3.5}$$

For the differentials (2.3), *for* Ω *above, and for two variable points* (x,y) *and* (z,w) *on* C, *there exist second kind differentials* $\eta_j=\eta_j(x,y)$ $(j=1,\dots,4)$, *having their only pole at* ∞, *such that*

$$R\left((x,y),(z,w)\right):=\frac{\mathrm{d}}{\mathrm{d}x}\Omega\left((x,y),(z,w)\right)+\sum_{j=1}^{4}\frac{\omega_j(x,y)}{\mathrm{d}x}\frac{\eta_j(z,w)}{\mathrm{d}z}, \tag{3.6}$$

where the derivation[1] *is with respect to the variable point* $(x,y)\in C$. *We further require that it satisfies the symmetry condition*

$$R\left((x,y),(z,w)\right)=R\left((z,w),(x,y)\right). \tag{3.7}$$

Then the set of differentials $\{\eta_1,\eta_2,\eta_3,\eta_4\}$ *is determined uniquely modulo the space spanned by the* ω_js *of* (2.3). *The* 2*-form obtained above is called (Klein's) fundamental* 2*-form of the second kind.*

Proof. The 2-form

$$\frac{\mathrm{d}}{\mathrm{d}x}\Omega\left((x,y),(z,w)\right)\mathrm{d}z\mathrm{d}x, \tag{3.8}$$

considered as a function of (x,y), satisfies the condition on the poles; indeed one can check that (3.8) has only a second-order pole at $(x,y)=(z,w)$ whenever (z,w) is either an ordinary point or a Weierstrass point; at infinity the expansion (2.13) should be used. However, the form (3.8) has unwanted poles at infinity, when considered as a form in the (z,w)-variables. To restore the symmetry required in (3.7) we complement (3.8) by the second term to obtain (3.6) with differentials $\eta_j(z,w)$, these should have poles only as $(z,w)\to\infty$.

It is easily seen that the η_j above can be written as

$$\eta_j(x,y)=\frac{h_j(x,y)}{\frac{\partial}{\partial y}f(x,y)}\mathrm{d}x, \tag{3.9}$$

where $h_j(x,y)\in\mathbf{Q}[\lambda_0,\dots,\lambda_4][[x,y]]$, and h_j is of homogeneous weight. Then the symmetry condition (3.7) results in a system of linear equations for the $h_j(z,w)$, which is always solvable. As a result, the polynomials $h_j(z,w)$ as well as $F((x,y),(z,w))$ are obtained explicitly. Straightforward calculations for the curve C Eq. (2.2) then lead to the following expressions:

$$\eta_1(x,y)=yx(3\lambda_3+7x^2+5x\lambda_4)\frac{\mathrm{d}x}{3x^2},$$
$$\eta_2(x,y)=2yx\,(2x+\lambda_4)\,\frac{\mathrm{d}x}{3x^2},$$

[1] Since x and y are related, we do not use ∂.

$$\eta_3(x, y) = (2x^3 + x^2\lambda_4 - \lambda_2)\frac{\mathrm{d}x}{3x^2},$$
$$\eta_4(x, y) = xy\frac{\mathrm{d}x}{3x^2}. \tag{3.10}$$

We now define the period matrices by

$$\left[\omega'\omega''\right] = \left[\int_{\alpha_i}\omega_j \int_{\beta_i}\omega_j\right]_{i,j=1,\dots,4}, \qquad \left[\eta'\eta''\right] = \left[\int_{\alpha_i}\eta_j \int_{\beta_i}\eta_j\right]_{i,j=1,\dots,4}. \tag{3.11}$$

We can combine these two matrices into

$$M = \begin{bmatrix}\omega' & \omega''\\ \eta' & \eta''\end{bmatrix}. \tag{3.12}$$

The matrix M then satisfies

$$M\begin{bmatrix} & -1_4\\ 1_4 & \end{bmatrix}{}^tM = 2\pi\sqrt{-1}\begin{bmatrix} & -1_4\\ 1_4 & \end{bmatrix}. \tag{3.13}$$

This is the *generalized Legendre relation* (see (1.14) on p. 11 of [6]). In particular, $\omega'^{-1}\omega''$ is a symmetric matrix. We know also that

$$\mathrm{Im}(\omega'^{-1}\omega'') \quad \text{is positive definite.} \tag{3.14}$$

Let

$$\delta := \begin{bmatrix}\delta'\\ \delta''\end{bmatrix} \in \left(\frac{1}{2}\mathbf{Z}\right)^8 \tag{3.15}$$

be the theta characteristic which gives the Riemann constant with respect to the base point ∞ and the period matrix $[\omega'\ \omega'']$ ([16], pp. 163–166, [6], p. 15, (1.18)). By looking at (2.3), we see the canonical divisor class of C is given by 4∞. Hence any theta characteristic is an element of $\left(\frac{1}{2}\mathbf{Z}\right)^8$ in this case.

We then define

$$\begin{aligned}\sigma(u) &= \sigma(u; M) = \sigma(u_1, u_2, u_3, u_4; M)\\ &= c\exp\left(-\frac{1}{2}u\eta'\omega'^{-1}\,{}^tu\right)\vartheta\,[\delta]\,(\omega'^{-1}\,{}^tu;\omega'^{-1}\omega'')\\ &= c\exp\left(-\frac{1}{2}u\eta'\omega'^{-1}\,{}^tu\right)\sum_{n\in\mathbf{Z}^4}\exp\left[2\pi i\left\{\frac{1}{2}\,{}^t(n+\delta')\omega'^{-1}\omega''(n+\delta') + {}^t(n+\delta')(z+\delta'')\right\}\right],\end{aligned} \tag{3.16}$$

where c is a constant depending only on the parameters of the curve, $\{\lambda_0, \dots, \lambda_4\}$, which we fix below. The series (3.16) converges for all $u \in \mathbf{C}^4$ because of property (3.14).

In what follows, for any given $u \in \mathbf{C}^4$, we denote by u' and u'' the unique elements in $\mathbf{R}^4$ such that

$$u = u'\omega' + u''\omega''. \tag{3.17}$$

Then for $u, v \in \mathbf{C}^4$, and $\ell\ (= \ell'\omega' + \ell''\omega'') \in \Lambda$, we define

$$L(u, v) := {}^tu(\eta'v' + \eta''v''),$$
$$\chi(\ell) := \exp[\pi\sqrt{-1}\left(2({}^t\ell'\,\delta'' - {}^t\ell''\,\delta') + {}^t\ell'\,\ell''\right)] \quad (\in \{1, -1\}). \tag{3.18}$$

The most important properties of $\sigma(u; M)$ can be expressed as follows.

Lemma 3.2. *For all $u \in \mathbf{C}^4$, $\ell \in \Lambda$, and $\gamma \in Sp(8, \mathbf{Z})$, we have:*

$$\sigma(u + \ell; M) = \chi(\ell)\sigma(u; M)\exp L\left(u + \frac{1}{2}\ell, \ell\right), \tag{3.19}$$

$$\sigma(u;\gamma M)=\sigma(u;M), \tag{3.20}$$

$$u\mapsto\sigma(u;M)\quad \textit{has zeroes of order } 1 \textit{ along } \Theta^{[3]}, \tag{3.21}$$

$$\sigma(u;M)=0 \iff u\in\Theta^{[3]}. \tag{3.22}$$

Proof. The formula (3.19) is a special case of the equation from [4] (p.286, ℓ.22). The statement (3.20) is easily shown by the definition of $\sigma(u)$ since γ corresponds to the choice of basis of cycles $\{\alpha_j,\beta_j\}_{j=1}^4$ (3.2), which are used to define the periods (3.11). The statements (3.21) and (3.22) are explained in [4], (p. 252). These facts are partially described also in [6], (p. 12, Th. 1.1 and p. 15). □

Remark 3.3. We fix a matrix M satisfying (3.13) and (3.14). The space of the solutions of (3.19) is a one dimensional space over $\mathbf{C}$, because the Pfaffian of the Riemann form attached to $L(.)$ is 1 (see [17], Lemma 3.1.2 and [15], p. 93, Th. 3.1). Hence, such non-trivial solutions automatically satisfy (3.20)–(3.22). In this sense, (3.19) characterizes the function $\sigma(u)$ up to a constant factor. As a corollary, since $\sigma(-u)$ also satisfies this condition, it follows that $\sigma(u)$ must have definite odd or even parity.

The constant c of (3.16) can be fixed as follows:

Lemma 3.4. *The power series expansion of $\sigma(u)$ about $u=(0,0,0,0)$ with respect to u_1, u_2, u_3 and u_4 has homogeneous Sato weight 8; its leading term is the Schur–Weierstrass polynomial $S(u)$ corresponding to the sequence of Sato weights of the $\{u_i\}$, which is $\{7,4,2,1\}$ for any $(3,5)$ curve. Explicitly:*

$$S(u)=\left(\frac{1}{448}u_4^8+u_2^2+u_2u_3u_4^2-\frac{1}{8}u_3^2u_4^4-\frac{1}{4}u_3^4-u_1u_4\right).$$

The expansion is then of the form

$$\sigma(u)=\left(\frac{1}{448}u_4^8+u_2^2+u_2u_3u_4^2-\frac{1}{8}u_3^2u_4^4-\frac{1}{4}u_3^4-u_1u_4\right)+(d^\circ(\lambda_0,\ldots,\lambda_4)\geqq 1).$$

In particular, by Remark 3.3, all the higher monomials in this expansion must also be even, so here $\sigma(u)$ is an even function.

Proof. See [20]. The essential part of this assertion is seen also by [7]. □

In addition to the σ-function, we may also define N-th-order theta functions on $\mathbf{C}^4$.

Definition 3.5. An N-th-order theta function is any function $f(u)$ on $\mathbf{C}^4$ satisfying the same periodicity condition as $\sigma(u)^N$, that is:

$$f(u+\ell)=\chi(\ell)^N f(u)\exp\left(NL\left(u+\frac{1}{2}\ell,\ell\right)\right). \tag{3.23}$$

Such functions will be used to construct Abelian functions below.

4. Abelian functions

Definition 4.1. A meromorphic function $\mathfrak{P}(u)$ is called an Abelian function of $u\in\mathbb{C}^4$, with respect to the period lattice Λ with generators ω' and ω'', if it is multiply periodic, that is, if

$$\mathfrak{P}(u+\omega' n^T+\omega'' m^T)=\mathfrak{P}(u) \tag{4.1}$$

for all integer vectors $n,m\in\mathbb{Z}$ wherever $\mathfrak{P}(u)$ exists.

We note from Definition 3.5 that the quotient of two N-th-order theta functions must be Abelian.

To construct Abelian functions in terms of the σ-functions, we first note that

$$\sigma(u+v)\sigma(u-v)$$

is a second-order theta function in u. We then take derivatives with respect to the parameter v, denoting:

$$\Delta_i = \frac{\partial}{\partial v_i} \tag{4.2}$$

for $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$; then we define a set of fundamental Abelian functions on J by

$$\wp_{ij}(u) = -\frac{1}{2\sigma(u)^2}\Delta_i\Delta_j\sigma(u+v)\sigma(u-v)|_{v=0} = -\frac{\partial^2}{\partial u_i \partial u_j}\log\sigma(u). \tag{4.3}$$

Evidently these functions are singular where $\sigma(u) = 0$; this is on the set $\Theta^{[3]}$.

For the benefit of the reader familiar only with the genus 1 case, we should point out that the Weierstrass function $\wp(u)$ described in Eq. (1.3) would be written as $\wp_{11}(u)$ in this notation. Moreover, we define

$$\wp_{ijk}(u) = \frac{\partial}{\partial u_k}\wp_{ij}(u), \qquad \wp_{ijk\ell}(u) = \frac{\partial}{\partial u_\ell}\wp_{ijk}(u), \tag{4.4}$$

and so on for higher derivatives. The functions (4.3) and (4.4) are periodic functions because of (3.19). Moreover, following Baker [5] and as generalized in [14], we define

$$\begin{aligned} Q_{ijk\ell}(u) &= -\frac{1}{2\sigma(u)^2}\Delta_i\Delta_j\Delta_k\Delta_\ell\sigma(u+v)\sigma(u-v)|_{v=0} \\ &= \wp_{ijk\ell}(u) - 2(\wp_{ij}\wp_{k\ell} + \wp_{ik}\wp_{j\ell} + \wp_{i\ell}\wp_{jk})(u). \end{aligned} \tag{4.5}$$

A short calculation shows that

$$Q_{ijk\ell} \in \Gamma(J, \mathcal{O}(2\Theta^{[3]})),$$

that is the vector space of meromorphic functions having at worst a double pole where $\sigma = 0$, but

$$\wp_{ijk\ell} \in \Gamma(J, \mathcal{O}(4\Theta^{[3]})),$$

having instead quadruple poles on the same set. Indeed, we see further that any expression of the Hirota form

$$Q_{ijk\ell mn}(u) = -\frac{1}{2\sigma(u)^2}\Delta_i\Delta_j\Delta_k\Delta_\ell\Delta_m\Delta_n\sigma(u+v)\sigma(u-v)|_{v=0}, \tag{4.6}$$

is an element of $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$, as is any similar expression of higher even order in the Δ_i; we note that such expressions of odd order necessarily vanish. Further, these may all be expressed as polynomials in the Kleinian $\wp_{ij}$ and their derivatives.

Note that although the subscripts in $\wp_{ijk\ell}$ *do* denote differentiation, the subscripts in $Q_{ijk\ell}$ do *not* denote direct differentiation, and the latter notation is introduced for convenience only. This is important to bear in mind when we use cross-differentiation, for example the $\wp_{ijk\ell}$ satisfy

$$\frac{\partial}{\partial u_m}\wp_{ijk\ell}(u) = \frac{\partial}{\partial u_\ell}\wp_{ijkm}(u),$$

whereas the $Q_{ijk\ell}$ do not.

The following formula involving the fundamental Kleinian $\wp$-functions was derived in [8] and was evaluated for the case of the curve (2.3) in [9]. Here it is noted that the unique Klein bidifferential may be written in two different ways, either in terms of the second derivatives of $\ln(\sigma)$, or else in terms of rational functions of the coordinates of points on the curve.

Theorem 4.1. *For arbitrary (x, y), and base point ∞ on C, an arbitrary set S of $g = 4$ distinct points $\{(x_1, y_1), \ldots, (x_4, y_4)\} \in C^4$, and (z, w) being any point of S, it follows that*

$$\sum_{i,j=1}^{4}\wp_{i,j}\left(\int_\infty^t \mathbf{du} - \sum_{k=1}^{4}\int_\infty^{x_k}\mathbf{du}\right)\mathcal{U}_i(x, y)\mathcal{U}_j(z, w) = \frac{F(x, y; z, w)}{(x-z)^2}, \tag{4.7}$$

where

$$\mathcal{U}^{\mathrm{T}}(x, y) = (1, x, y, x^2)$$

the vector of numerators of the ω_i, *and* F *is the symmetric function*

$$\begin{aligned} &F(x, y; z, w) \\ &\quad = 3w^2y^2 + [2z^3x^2 + z^4x + 3\lambda_0 + \lambda_1(2z + x) + \lambda_2(z^2 + 2xz) + \lambda_3(3z^2x) + \lambda_4(2z^3x + x^2z^2)]y \\ &\qquad + [2x^3z^2 + x^4z + 3\lambda_0 + \lambda_1(2x + z) + \lambda_2(x^2 + 2xz) + \lambda_3(3x^2z) + \lambda_4(2x^3z + x^2z^2)]w \end{aligned}$$

which appears in the numerator of the second kind fundamental 2*-form:*

$$\frac{F(x, y; z, w)}{(x - z)^2} \frac{\mathrm{d}x}{f_y(x, y)} \frac{\mathrm{d}z}{f_w(z, w)} = \mathrm{d}\Omega(x, y; z, w)$$

while $f(z, w) = 0$ *is the equation of the curve* C*:*

$$f(z, w) = w^3 - (z^5 + \lambda_4z^4 + \lambda_3z^3 + \lambda_2z^2 + \lambda_1z + \lambda_0).$$

Expanding (4.7) as one of the x_i tends to infinity and comparing the principal parts of the poles on both sides of the relation, we find, in leading order, the solution of the Jacobi inversion problem, first given explicitly for this curve in [9], following [13].

Theorem 4.2 (*Jacobi Inversion Formula [[6], p. 32] [9]*). *Let* C *be the genus* 4 *cyclic* (3, 5) *curve. If* $D = ((x_1, y_1) + (x_2, y_2) + (x_3, y_3) + (x_4, y_4))$ *is a non-special divisor,* d**u** *is the vector of holomorphic differentials, with period lattice* Λ*, then the Abel map is given by:*

$$u = \sum_{k=1}^{4} \int_{\infty}^{x_k} \mathrm{d}\mathbf{u} \mod \Lambda.$$

The Abel preimage of the point $u \in \mathbb{C}^4$ *is then given by the set*

$$\{(x_1, y_1), \ldots, (x_4, y_4)\} \in (C)^4,$$

where $\{x_1, \ldots, x_4\}$ *are the zeros of the polynomial*

$$\begin{aligned} \mathcal{P}(x, y; u) = {} & 2x^4 + (\lambda_4 - \wp_{444} - 3\wp_{34})x^3 \\ & + \left(-\wp_{34}\lambda_4 - \wp_{44}\wp_{33} + \wp_{444}\wp_{34} + \wp_{34}^2 - \wp_{244} - \wp_{23} - \wp_{44}\wp_{344}\right) x^2 \\ & + (-\wp_{13} - \wp_{144} + \wp_{244}\wp_{34} + \wp_{23}\wp_{34} - \wp_{24}\wp_{344} - \wp_{24}\wp_{33})\, x \\ & + \wp_{13}\wp_{34} - \wp_{14}\wp_{344} - \wp_{14}\wp_{33} + \wp_{144}\wp_{34}, \end{aligned}$$

and the coordinate y_i *of each point in* D *is given by*

$$\wp_{14} + \wp_{24}x_i + \wp_{34}y_i + \wp_{44}x_i^2 - x_iy_i = 0. \tag{4.8}$$

Expanding both sides of (4.7) to higher order, we obtain successively higher order differential polynomials in the $\wp_{ij}$, which must vanish identically. These play an analogous role in the theory to Weierstrass' fundamental differential equation for the elliptic $\wp$, (1.3). One of these was given in [9], but more recent arguments based on the parity of σ allow such equations to be greatly simplified. One obtains sets of relations of third order, either linear or quadratic in the $\wp_{ijk}$, and others, of fourth order, linear in the $\wp_{ijkl}$.

The first few of the latter include:

$$Q_{4444} = -3\wp_{33}, \tag{4.9}$$

$$Q_{3444} = 3\wp_{24}, \tag{4.10}$$

$$Q_{2344} = 4\wp_{14} - \wp_{22} + 2\lambda_4\wp_{24}. \tag{4.11}$$

$$Q_{1344} = -\wp_{12} + 2\lambda_4\wp_{14}. \tag{4.12}$$

The simplest two quadratic relations are:

$$\wp_{444}^2 = 4\wp_{44}^3 - 4\wp_{44}\wp_{33} + \wp_{34}^2 - 4\wp_{23} + 2\lambda_4\wp_{34} + \lambda_4^2 - 4\lambda_3, \tag{4.13}$$

$$\wp_{344}^2 = 4\wp_{34}^2\wp_{44} + 4\wp_{24}\wp_{34} + \wp_{33}^2 + 4\wp_{14}. \tag{4.14}$$

Lists of all known relations of each class are given in Section 7.

5. A basis of the space $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$

The vector space $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$ has dimension $2^g = 2^4 = 16$. All elements $\mathcal{V}(u)$ of this space may be written as

$$\mathcal{V}(u) = \frac{f(u)}{\sigma(u)^2}; \tag{5.1}$$

here $\mathcal{V}(u)$ is an Abelian function and $f(u)$ is a second-order theta function.

We note that second-order theta functions with this period lattice form a vector space, which by (5.1) is isomorphic to $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$.

Lemma 5.1. *We have the following basis*

$$\begin{aligned}\Gamma(J, \mathcal{O}(2\Theta^{[3]})) = {} & \mathbf{C}Q_{1144} \oplus \mathbf{C}\wp_{11} \oplus \mathbf{C}Q_{1244} \oplus \mathbf{C}Q_{2233} \oplus \mathbf{C}\wp_{12} \oplus \mathbf{C}Q_{1444} \oplus \mathbf{C}\wp_{13} \oplus \mathbf{C}\wp_{14} \\ & \oplus \mathbf{C}\wp_{22} \oplus \mathbf{C}Q_{2444} \oplus \mathbf{C}\wp_{23} \oplus \mathbf{C}\wp_{24} \oplus \mathbf{C}\wp_{33} \oplus \mathbf{C}\wp_{34} \oplus \mathbf{C}\wp_{44} \oplus \mathbf{C}1.\end{aligned}$$

Proof. We know the dimension of the space $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$ is $2^4 = 16$, by the Riemann–Roch theorem for Abelian varieties (see for example, [16], (pp. 150–155), [15], (p. 99, Th. 4.1)). Obviously, (4.3) and (4.5) show that the functions on the right hand sides each belong to the space on the left hand side. It thus only remains to verify their linear independence, and it is sufficient to do this in the special case where all the $\lambda_j = 0$, in which case $\sigma(u)$ reduces to the Schur–Weierstrass polynomial. We multiply each of the functions on the right hand side by $\sigma(u)^2$; this yields in each case a polynomial in (u_1, u_2, u_3, u_4). Then the functions on the right hand side are linearly independent; for all of these polynomials have different Sato weights, except for those from $\wp_{22}$ and $\wp_{14}$, and it is easy to check the independence of these two directly. $\square$

Corollary 5.2. *A second-order theta function whose Taylor expansion at* $(0, 0, 0, 0)$ *has a leading term of Sato weight* 16*, is a multiple of* $\sigma(u)^2$.

Proof. A second-order theta function must be a linear combination of $\sigma(u)^2$ with other basis elements, all of Sato weight less than 16. These must be absent if the leading order is 16. $\square$

Corollary 5.3. *A second-order theta function whose Taylor expansion at* $(0, 0, 0, 0)$ *has a leading term of Sato weight greater than* 16*, is identically zero.*

Proof. No non-trivial linear combination of these basis elements has leading weight greater than 16. $\square$

6. Expansion of the σ-function

We need more terms of the power series expansion of $\sigma(u)$, beyond those given in [9], in order to properly characterize the curve. The terms of Sato weight 23 in u_i are needed for this, as these are the first containing any dependence on λ_0.

Lemma 6.1. *The function* $\sigma(u)$ *associated with* (2.2) *has an expansion of the following form:*

$$\begin{aligned}\sigma(u_1, u_2, u_3, u_4) = {} & C_8(u_1, u_2, u_3, u_4) + C_{11}(u_1, u_2, u_3, u_4) + C_{14}(u_1, u_2, u_3, u_4) + C_{17}(u_1, u_2, u_3, u_4) \\ & + C_{20}(u_1, u_2, u_3, u_4) + C_{23}(u_1, u_2, u_3, u_4) + \cdots + C_{5+3n}(u_1, u_2, u_3) + \cdots,\end{aligned} \tag{6.1}$$

where each C_{5+3n} *is an even polynomial in the* u_i*, composed of products of monomials in* u_i *of total weight* $5 + 3n$ *multiplied by monomials in* λ_i *of total weight* $-3n$*. The first few* C_k *are given in* [9].

Proof. The proof is by construction (with heavy use of Maple) initially following the methods used in [9]. In particular, we require C_8 to be the Schur–Weierstrass polynomial, and then most of the higher terms are fixed by requiring the vanishing of the sigma function on the strata $\Theta^{[1]}$, $\Theta^{[2]}$ and $\Theta^{[3]}$. This determines σ up to multiplication by an even analytic function equal to 1 at the origin. In addition we fix the remaining coefficients up to C_{23} inclusive, by requiring that σ satisfies (through the definitions (4.3) and (4.5)) the two equations:

$$Q_{4444} = -3\wp_{33}, \tag{6.2}$$
$$Q_{1344} = -\wp_{12} + 2\lambda_4\wp_{14}. \tag{6.3}$$

These equations, we recall, were found by expanding the Klein bidifferential and equating coefficients. □

Remark 6.2. Other relations of this form, in which linear combinations of the $Q_{ijk\ell}$ are expressed as linear combinations of the $\wp_{ij}$ could in principle be found by expanding the Klein bidifferential to (much) higher order, but it is much easier to derive them directly from the expansion of σ, as will be done below.

7. Equations satisfied by the Abelian functions associated with *C*

We can use the σ-function expansion to identify various further equations which the Abelian functions defined by (4.3) and (4.5) must satisfy

7.1. Four-index relations

Such relations are the generalizations of $\wp'' = 6\wp^2 - \frac{1}{2}g_2\wp$ in the cubic (genus 1) case.

Proposition 7.1. *The four-index functions $Q_{ijk\ell}$ associated with (2.2) satisfy the following relations*

$$\begin{aligned}
Q_{4444} &= -3\wp_{33},\\
Q_{3444} &= 3\wp_{24},\\
Q_{3344} &= -\wp_{23} + 2\lambda_4\wp_{34},\\
Q_{3334} &= -Q_{2444},\\
Q_{2344} &= 4\wp_{14} - \wp_{22} + 2\lambda_4\wp_{24},\\
Q_{3333} &= 12\wp_{14} - 3\wp_{22},\\
Q_{2334} &= \lambda_2 + 2\wp_{13} + 3\lambda_3\wp_{34}\\
Q_{1444} &= -\frac{1}{2}Q_{2333} + \frac{3}{2}\lambda_3\wp_{33},\\
Q_{2244} &= -\frac{1}{3}Q_{2333} - \frac{2}{3}\lambda_4 Q_{3334} + 2\lambda_3\wp_{33},\\
Q_{1344} &= 2\lambda_4\wp_{14} - \wp_{12},\\
Q_{2234} &= -2\wp_{12} + 4\lambda_4\wp_{14} + 3\lambda_3\wp_{24} - 2\lambda_2\wp_{44},\\
Q_{1334} &= -\frac{1}{2}Q_{2233} + 2\lambda_4\wp_{13} + \frac{3}{2}\lambda_3\wp_{23} + 2\lambda_2\wp_{34} + \lambda_4\lambda_2,\\
Q_{1333} &= 3Q_{1244} + \lambda_4 Q_{2333} - 3\lambda_4\lambda_3\wp_{33},\\
Q_{1234} &= -\wp_{11} + 3\lambda_3\wp_{14} - \lambda_1\wp_{44},\\
Q_{2223} &= -6\wp_{11} + 6\lambda_3\wp_{14} + 3\lambda_3\wp_{22} - 6\lambda_1\wp_{44},\\
Q_{1233} &= -3\lambda_0 + \lambda_4\lambda_1 + 3\lambda_3\wp_{13} + 2\lambda_1\wp_{34},\\
Q_{1144} &= -Q_{1224} - \frac{1}{2}\lambda_3 Q_{2333} + \frac{3}{2}\lambda_3^2\wp_{33} + 3\lambda_1\wp_{33},\\
Q_{1134} &= 2\wp_{14}\lambda_2 - \wp_{24}\lambda_1\\
Q_{1223} &= -2\lambda_4\wp_{11} + 3\lambda_3\wp_{12} + 4\lambda_2\wp_{14} - 2\lambda_1\wp_{24} - 6\lambda_0\wp_{44},
\end{aligned}$$

$$Q_{1222} = 6(\lambda_0 + \lambda_4\lambda_1)\wp_{33} - \lambda_3 Q_{1333} + 2\lambda_1 Q_{2444},$$
$$\ldots.$$

Proof. Since any four-index function $Q_{ijk\ell}$ belongs to $\Gamma(J, \mathcal{O}(2\Theta^{[3]}))$, it must be in the span of the basis (5.1) — hence a relation must hold of the form:

$$Q_{ijk\ell} = \text{ linear function of various } \wp_{mn} \text{ and other } Q_{opqr},$$

with λ-dependent coefficients. Thus, it is just a matter of enumerating all the possible terms on the right hand side which have the same weight as $Q_{ijk\ell}$. Using the method of undetermined coefficients, we insert the σ expansion truncated to an appropriate weight in the λ_i and solve for the unknown (constant) coefficients. This requires the use of Maple but is quite efficient as the pole on either side is only of order 2 in σ. $\square$

Remark 7.2. For completeness we give the explicit formulae for the Q in terms of the $\wp$

$$Q_{ijkk} = \wp_{ijkk} - 2\wp_{ij}\wp_{kk} - 4\wp_{ik}\wp_{jk}, \qquad Q_{iikk} = \wp_{iikk} - 2\wp_{ii}\wp_{kk} - 4\wp_{ik}^2,$$
$$Q_{ikkk} = \wp_{ikkk} - 6\wp_{ik}\wp_{kk}, \qquad Q_{kkkk} = \wp_{kkkk} - 6\wp_{kk}^2.$$

Remark 7.3. The complete set of such relations for the cyclic (3, 4) curve was given in [14]. As far as we know, the above incomplete set of relations for the cyclic (3, 5) curve is new, and a comparison is of interest.

Remark 7.4. The first equation in the list given in Proposition 7.1 above is:

$$\wp_{4444} = 6\wp_{44}^2 - 3\wp_{33}.$$

This relation, after differentiating twice with respect to u_4, becomes the Boussinesq equation for $\wp_{44}$; u_4 and u_3 respectively play the roles of the space and time variables here. The connection between the Boussinesq equation and cyclic trigonal curves is well established (see [8,13]).

7.2. Linear three-index relations

Proposition 7.5. *The three-index functions $\wp_{ijk}$ associated with (2.2) satisfy a number of relations linear in these functions. These have no analogue in the genus 1 case. For example in decreasing weight, starting at -6, we have*

$$\begin{aligned}
\wp_{333} &= 2\wp_{44}\wp_{344} - 2\wp_{34}\wp_{444} - \wp_{244},\\
\wp_{234} &= \frac{1}{2}\wp_{34}\wp_{344} - \wp_{334}\wp_{44} + \frac{1}{2}\wp_{33}\wp_{444} + \frac{1}{2}\lambda_4\wp_{344},\\
\wp_{233} &= -\wp_{33}\wp_{344} - \frac{3}{2}\wp_{444}\wp_{24} + \frac{1}{2}\wp_{334}\wp_{34} + \frac{3}{2}\wp_{244}\wp_{44} + \frac{1}{2}\lambda_4\wp_{334} + \frac{1}{2}\wp_{333}\wp_{44},\\
\wp_{144} &= -\frac{1}{2}\wp_{334}\wp_{33} + \frac{1}{2}\wp_{333}\wp_{34} + \wp_{344}\wp_{24} - \frac{1}{2}\wp_{34}\wp_{244},\\
\wp_{134} &= \wp_{234}\wp_{34} - \wp_{24}\wp_{334} + \frac{1}{2}\wp_{33}\wp_{244} - \frac{1}{2}\wp_{344}\wp_{23},\\
\wp_{133} &= \frac{1}{2}\wp_{333}\wp_{24} - \wp_{33}\wp_{234} - \frac{1}{2}\wp_{23}\wp_{334} + \wp_{34}\wp_{233} - 3\wp_{444}\wp_{14} + 3\wp_{144}\wp_{44},\\
\wp_{124} &= -\wp_{134}\wp_{44} - \frac{1}{2}\wp_{144}\wp_{34} + \wp_{14}\wp_{344} + \frac{1}{2}\wp_{13}\wp_{444} + \frac{1}{2}\lambda_4\wp_{144},\\
\wp_{134}\wp_{34} &= -\frac{1}{2}\wp_{33}\wp_{144} + \frac{1}{2}\wp_{344}\wp_{13} + \wp_{334}\wp_{14},\\
\wp_{114} &= -\frac{1}{2}\wp_{144}\wp_{23} - \wp_{134}\wp_{24} + \wp_{234}\wp_{14} + \frac{1}{2}\wp_{244}\wp_{13},\\
\wp_{111} &= \frac{2}{3}\wp_{22}\wp_{123} + \frac{1}{3}\wp_{23}\wp_{122} + \lambda_3\wp_{114} - \lambda_1\wp_{144} - \frac{1}{3}\wp_{13}\wp_{222}\\
&\quad - \frac{2}{3}\wp_{223}\wp_{12} - \frac{2}{3}\lambda_2\wp_{124} + \frac{1}{3}\lambda_1\wp_{224} + \lambda_0\wp_{244} + \frac{1}{3}\lambda_4\wp_{112}.
\end{aligned}$$

Proof. These can be calculated directly by expressing the equations in Proposition 7.1 in terms of $\wp_{ijk\ell}$ and $\wp_{mn}$ functions, then using cross-differentiation on suitably chosen pairs of equations. For example, the first relation above for $\wp_{333}$ comes from

$$\frac{\partial}{\partial u_3}\wp_{4444} - \frac{\partial}{\partial u_4}\wp_{3444} = 0.$$

In principle the relations could be checked by inserting the σ expansion to the required level, but this is method is much slower and more cumbersome since the terms involve poles of order 3. □

7.3. Quadratic three-index relations

Proposition 7.6. *Quadratic expressions in the three-index functions $\wp_{ijk}$ associated with (2.2) can be expressed in terms of (at most cubic) relations in the $\wp_{mn}$ and $\wp_{ijk\ell}$. For example we have the following*

$$\begin{aligned}
\wp_{444}^2 &= 4\wp_{44}^3 - 4\wp_{44}\wp_{33} + \wp_{34}^2 - 4\wp_{23} + 2\lambda_4\wp_{34} + \lambda_4^2 - 4\lambda_3,\\
\wp_{344}\wp_{444} &= 4\wp_{34}\wp_{44}^2 + 6\wp_{24}\wp_{44} - \wp_{33}\wp_{34} - \wp_{33}\lambda_4 - \frac{2}{3}\wp_{2444},\\
\wp_{344}^2 &= 4\wp_{34}^2\wp_{44} + 4\wp_{24}\wp_{34} + \wp_{33}^2 + 4\wp_{14},\\
\wp_{334}\wp_{444} &= 2\wp_{34}^2\wp_{44} - \wp_{24}\wp_{34} - 2\wp_{33}^2 - 4\wp_{14} - 2\wp_{44}\wp_{23} + 2\wp_{22} - \wp_{24}\lambda_4 + 2\wp_{44}\lambda_4\wp_{34} + 2\wp_{33}\wp_{44}^2,\\
\wp_{334}\wp_{344} &= 2\wp_{44}\wp_{33}\wp_{34} + 2\wp_{34}^3 - 2\wp_{23}\wp_{34} + \wp_{24}\wp_{33} - 2\wp_{13} + 2\lambda_4\wp_{34}^2,\\
\wp_{333}\wp_{444} &= 6\wp_{44}\wp_{33}\wp_{34} - 2\wp_{34}^3 + 7\wp_{34}\wp_{23} + 2\wp_{33}\wp_{24} + 2\wp_{13} - 4\lambda_4\wp_{34}^2\\
&\quad + 2\lambda_2 + 6\wp_{34}\lambda_3 + \lambda_4\wp_{23} - 2\lambda_4^2\wp_{34} - 2\lambda_4\wp_{44}\wp_{33} + 12\wp_{24}\wp_{44}^2 - 2\wp_{44}\wp_{2444},\\
\wp_{444}\wp_{244} &= \frac{2}{3}\wp_{44}\wp_{2444} - 2\wp_{13} + \wp_{34}\wp_{23} - 2\wp_{33}\wp_{24} - 2\lambda_2 + 2\wp_{34}\lambda_3 - \lambda_4\wp_{23},\\
\wp_{334}^2 &= 4\wp_{33}\wp_{34}^2 + 8\wp_{34}\wp_{24}\wp_{44} - \frac{4}{3}\wp_{2444}\wp_{34} + \wp_{24}^2 - \frac{4}{3}\wp_{1444} + 4\wp_{44}\wp_{14},\\
\wp_{344}\wp_{333} &= 2\wp_{33}\wp_{34}^2 - \wp_{33}\wp_{23} + 2\wp_{33}\lambda_4\wp_{34} + 2\wp_{33}^2\wp_{44} - 4\wp_{34}\wp_{24}\wp_{44}\\
&\quad + \frac{2}{3}\wp_{2444}\wp_{34} - 2\wp_{24}^2 + \frac{2}{3}\wp_{1444} + 4\wp_{44}\wp_{14},\\
\wp_{344}\wp_{244} &= \wp_{33}\wp_{23} + \frac{2}{3}\wp_{2444}\wp_{34} + 2\wp_{24}^2 - \frac{2}{3}\wp_{1444} + 4\wp_{44}\wp_{14},\\
\wp_{444}\wp_{234} &= 4\wp_{34}\wp_{24}\wp_{44} - \frac{1}{3}\wp_{2444}\wp_{34} - 2\wp_{33}\wp_{23} + 4\lambda_4\wp_{24}\wp_{44} - \frac{1}{3}\lambda_4\wp_{2444}\\
&\quad - 2\wp_{33}\lambda_3 + 4\wp_{44}\wp_{14} + 2\wp_{23}\wp_{44}^2 - 2\wp_{22}\wp_{44},\\
\wp_{344}\wp_{234} &= 2\wp_{14}\lambda_4 + 2\wp_{44}\wp_{33}\wp_{24} + 2\wp_{34}\wp_{23}\wp_{44} + 2\wp_{44}\wp_{13} + 2\wp_{14}\wp_{34}\\
&\quad + 2\wp_{24}\wp_{34}^2 + 2\wp_{34}\lambda_4\wp_{24} - \frac{1}{3}\wp_{33}\wp_{2444},\\
\wp_{444}\wp_{233} &= 6\wp_{44}\wp_{34}\lambda_3 - 2\wp_{44}\lambda_2 + 4\wp_{34}\wp_{23}\wp_{44} - 4\wp_{34}\lambda_4\wp_{24} - 2\wp_{24}\wp_{34}^2\\
&\quad - 2\wp_{14}\wp_{34} + \wp_{34}\wp_{22} - 2\wp_{44}\wp_{13} - 2\wp_{44}\wp_{33}\wp_{24} + \lambda_4\wp_{22}\\
&\quad - 2\lambda_4\wp_{23}\wp_{44} + \frac{2}{3}\wp_{33}\wp_{2444} + 6\wp_{24}\wp_{23} - 2\wp_{24}\lambda_4^2 + 6\wp_{24}\lambda_3 - 2\wp_{14}\lambda_4,\\
\wp_{334}\wp_{244} &= 2\wp_{34}\lambda_4\wp_{24} + 2\wp_{24}\wp_{34}^2 + 2\wp_{14}\wp_{34} - 2\wp_{34}\wp_{22} + 2\wp_{12} - 2\wp_{44}\wp_{13}\\
&\quad - 2\wp_{44}\wp_{33}\wp_{24} + \frac{2}{3}\wp_{33}\wp_{2444} - \wp_{24}\wp_{23} - 2\wp_{14}\lambda_4,\\
\wp_{334}\wp_{333} &= 6\wp_{14}\wp_{34} - 2\wp_{34}\wp_{22} - 2\wp_{12} - 2\wp_{44}\wp_{13} + 4\wp_{44}\wp_{33}\wp_{24}\\
&\quad - \frac{2}{3}\wp_{33}\wp_{2444} + \wp_{24}\wp_{23} + 2\wp_{14}\lambda_4 + 4\wp_{34}\wp_{33}^2,\\
\wp_{333}^2 &= 2\wp_{2233} - 4\lambda_4\lambda_2 + 4\lambda_1 - 8\wp_{33}\wp_{22} - 4\wp_{34}\wp_{13} - 6\wp_{23}\lambda_3\\
&\quad - 4\wp_{34}\lambda_2 - 4\lambda_4\wp_{13} - 7\wp_{23}^2 + 16\wp_{14}\wp_{33} + 4\wp_{33}^3,
\end{aligned}$$

$$\begin{aligned}
\wp_{444}\wp_{144} &= \frac{2}{3}\wp_{33}\wp_{22} + \wp_{34}\wp_{13} + \frac{2}{3}\wp_{44}\wp_{1444} + \frac{4}{3}\wp_{23}^2 - 2\wp_{33}\wp_{14} - \frac{1}{3}\wp_{2233} \\
&\quad + \frac{1}{3}\lambda_4\wp_{13} + \wp_{23}\lambda_3 + \frac{4}{3}\wp_{34}\lambda_2 - \frac{4}{3}\lambda_1 + \frac{2}{3}\lambda_4\lambda_2, \\
\wp_{244}^2 &= -4\wp_{44}\wp_{24}^2 - \frac{4}{3}\wp_{33}\wp_{22} - \frac{5}{3}\wp_{23}^2 + \frac{4}{3}\wp_{24}\wp_{2444} + \frac{2}{3}\wp_{2233} - \frac{8}{3}\lambda_4\wp_{13} \\
&\quad - 2\wp_{23}\lambda_3 + \frac{4}{3}\wp_{34}\lambda_2 - \frac{4}{3}\lambda_1 - \frac{4}{3}\lambda_4\lambda_2, \\
\wp_{244}\wp_{333} &= -\frac{4}{3}\wp_{24}\wp_{2444} + \left(\frac{8}{3}\wp_{13} + 2\wp_{34}\wp_{23}\right)\lambda_4 - \frac{2}{3}\wp_{2233} + \frac{5}{3}\wp_{23}^2 + \frac{4}{3}\wp_{33}\wp_{22} \\
&\quad + 4\wp_{34}\wp_{13} + 4\wp_{34}\wp_{33}\wp_{24} - 2\wp_{34}^2\wp_{23} - \frac{4}{3}\wp_{44}\wp_{1444} + 8\wp_{44}\wp_{24}^2 \\
&\quad + 2\wp_{44}\wp_{33}\wp_{23} + 8\wp_{14}\wp_{44}^2 + 2\wp_{23}\lambda_3 + \frac{8}{3}\wp_{34}\lambda_2 + \frac{4}{3}\lambda_1 + \frac{4}{3}\lambda_4\lambda_2 - 4\wp_{34}^2\lambda_3, \\
\wp_{233}\wp_{344} &= \frac{2}{3}\wp_{24}\wp_{2444} + \frac{1}{3}\wp_{2233} - \frac{4}{3}\wp_{23}^2 - \frac{5}{3}\wp_{33}\wp_{22} - \wp_{23}\lambda_3 + \frac{2}{3}\wp_{34}\lambda_2 \\
&\quad + 2\wp_{34}^2\wp_{23} - \frac{4}{3}\wp_{44}\wp_{1444} - 4\wp_{44}\wp_{24}^2 + 2\wp_{44}\wp_{33}\wp_{23} + 2\wp_{34}^2\lambda_3 \\
&\quad + \left(2\wp_{33}\wp_{24} - \frac{4}{3}\wp_{13}\right)\lambda_4 + 8\wp_{14}\wp_{44}^2 + \frac{4}{3}\lambda_1 - \frac{2}{3}\lambda_4\lambda_2, \\
\wp_{234}\wp_{334} &= -\frac{1}{3}\wp_{24}\wp_{2444} + \frac{1}{3}\wp_{2233} - \frac{4}{3}\wp_{23}^2 - \frac{2}{3}\wp_{33}\wp_{22} + 2\wp_{33}\wp_{14} - \wp_{23}\lambda_3 \\
&\quad + \frac{2}{3}\wp_{34}\lambda_2 - \frac{4}{3}\lambda_4\wp_{13} + 2\wp_{34}\wp_{33}\wp_{24} + 2\wp_{34}^2\wp_{23} + \frac{2}{3}\wp_{44}\wp_{1444} \\
&\quad + 2\wp_{44}\wp_{24}^2 + 2\wp_{34}^2\lambda_3 - 4\wp_{14}\wp_{44}^2 + \frac{4}{3}\lambda_1 - \frac{2}{3}\lambda_4\lambda_2, \\
\wp_{224}\wp_{444} &= -\frac{4}{3}\lambda_1 - \frac{1}{3}\lambda_4\lambda_2 + 2\wp_{34}\lambda_4\wp_{23} + \lambda_4\wp_{34}\lambda_3 + 2\wp_{44}\wp_{33}\lambda_3 - 2\lambda_4\wp_{33}\wp_{24} + 2\wp_{22}\wp_{44}^2 \\
&\quad + \frac{2}{3}\wp_{44}\lambda_4\wp_{2444} - 4\lambda_4\wp_{24}\wp_{44}^2 - \frac{2}{3}\wp_{23}^2 - 2\wp_{33}\wp_{14} + 2\wp_{34}\wp_{13} + \frac{2}{3}\wp_{44}\wp_{1444} - \wp_{34}^2\lambda_3 \\
&\quad - 4\wp_{14}\wp_{44}^2 - \frac{1}{3}\wp_{2233} + 6\wp_{44}\wp_{24}^2 + \frac{2}{3}\wp_{33}\wp_{22} - \frac{2}{3}\wp_{24}\wp_{2444} - \frac{2}{3}\lambda_4\wp_{13} - \wp_{23}\lambda_3 + \frac{7}{3}\wp_{34}\lambda_2, \\
\wp_{144}\wp_{344} &= \frac{2}{3}\wp_{1444}\wp_{34} + \wp_{13}\wp_{33} + 2\wp_{14}\wp_{24}.
\end{aligned}$$

Proof. Such relations can be found in three different ways. One is to multiply one of the linear three-index $\wp_{ijk}$ relations above by another $\wp_{ijk}$ and substitute for previously calculated $\wp_{ijk}\wp_{\ell mn}$ relations of higher weight. A second approach is to take a derivative of one of the linear three-index $\wp_{ijk}$ relations above and to substitute the known linear four-index $\wp_{ijk\ell}$ and previously calculated $\wp_{ijk}\wp_{\ell mn}$ relations. The third method is to use the method of undetermined coefficients: write down all possible terms which are cubic or less in the basis functions (5.1) of the correct weight. This last method is guaranteed to work but is much more time-consuming than the first two methods (when they work). At weight -15 this method requires some use of Distributed Maple on a multi-processor cluster. □

Remark 7.7. These relations are the generalizations of the familiar result $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ in the genus 1 theory.

Remark 7.8. It is evident that the left hand sides of the above relations are not independent. For instance from the first three of them, we may find two different expressions for $\wp_{444}^2\wp_{344}^2$. Identifying these we obtain a quadratic four-index relation:

$$\begin{aligned}
&(4\wp_{44}^3 - 4\wp_{44}\wp_{33} + \wp_{34}^2 - 4\wp_{23} + 2\lambda_4\wp_{34} + \lambda_4^2 - 4\lambda_3)(4\wp_{34}^2\wp_{44} + 4\wp_{24}\wp_{34} + \wp_{33}^2 + 4\wp_{14}) \\
&\quad - (4\wp_{34}\wp_{44}^2 + 6\wp_{24}\wp_{44} - \wp_{33}\wp_{34} - \wp_{33}\lambda_4 - \frac{2}{3}\wp_{2444})^2 = 0.
\end{aligned}$$

We note that the linear four-index relations above do not include one involving $\wp_{2444}$ alone.

8. The two-term addition theorem

Theorem 8.1. *The sigma function associated with (2.2) satisfies the following two-term addition formula:*

$$\begin{aligned}\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} &= -\wp_{11}(u)\wp_{44}(v)+\wp_{12}(u)\wp_{24}(v)-\frac{3}{4}\wp_{14}(u)\wp_{22}(v)\\ &+\frac{1}{3}\wp_{13}(u)Q_{2444}(v)+\frac{1}{12}\wp_{14}(u)Q_{3333}(v)+\frac{1}{6}\wp_{23}(u)Q_{2333}(v)+\frac{1}{3}\wp_{33}(u)Q_{1334}(v)\\ &-\frac{1}{3}\wp_{34}(u)Q_{1333}(v)-\frac{1}{12}Q_{2222}(u)-\frac{1}{3}\lambda_4 Q_{1333}(u)+\frac{1}{6}\lambda_3 Q_{2333}(u)\\ &-\frac{1}{2}\lambda_3\wp_{23}(u)\wp_{33}(v)+\frac{1}{3}\lambda_2 Q_{2444}(u)+\left(\frac{1}{3}\lambda_1+\lambda_4\lambda_2-\frac{3}{4}\lambda_3^2\right)\wp_{33}(u)+(u\Leftrightarrow v).\end{aligned} \tag{8.1}$$

Proof. Firstly, we notice that the left hand side is an even function with respect to $(u, v) \mapsto ([-1]u, [-1]v)$, and is a symmetric Abelian function in u and v. We note that it has poles of order 2 along $(\Theta^{[3]} \times J) \cup (J \times \Theta^{[3]})$ but nowhere else. Moreover it is of Sato weight -16. It is thus expressible as a symmetric bilinear combination of the basis functions in (5.1), with each pair of functions having total weight $-16 + 3n$, where the coefficients are either absolute constants, (if $n = 0$) or else are homogeneous polynomials (of weight $-3n$) in the λ_i. These undetermined coefficients were then found by substituting in the expansion of σ, and equating coefficients. □

Remark 8.2. By applying

$$\frac{1}{2}\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}+\frac{\partial}{\partial v_j}\right)\log \tag{8.2}$$

to (8.1), we have $\wp_{ij}(u+v)-\wp_{ij}(u)$ on the left hand side, and have a rational expression of several $\wp_{ij\cdots\ell}(u)$s and $\wp_{ij\cdots\ell}(v)$s on the right hand side. Hence, we have algebraic addition formulae for $\wp_{ij}(u)$s.

Remark 8.3. We note that the left hand side of the addition formula has a zero wherever $u = v$. By putting $v = u - w$ and letting $w \to (0, 0, 0, 0)$, the left hand side has leading term

$$\frac{\sigma(u+v)}{\sigma(u)^2\sigma(v)^2}(w_2^2-w_1w_4)+O(w^3).$$

On expanding the left and right hand sides, we can match powers of w, getting at zeroth order:

Corollary 8.4.

$$\begin{aligned}&-\wp_{11}(u)\wp_{44}(u)+\wp_{12}(u)\wp_{24}(u)-\frac{3}{4}\wp_{14}(u)\wp_{22}(u)+\frac{1}{3}\wp_{13}(u)Q_{2444}(u)\\ &\quad+\frac{1}{12}\wp_{14}(u)Q_{3333}(u)+\frac{1}{6}\wp_{23}(u)Q_{2333}(u)+\frac{1}{3}\wp_{33}(u)Q_{1334}(u)\\ &\quad-\frac{1}{3}\wp_{34}(u)Q_{1333}(u)-\frac{1}{12}Q_{2222}(u)-\frac{1}{3}\lambda_4 Q_{1333}(u)+\frac{1}{6}\lambda_3 Q_{2333}(u)\\ &\quad-\frac{1}{2}\lambda_3\wp_{23}(u)\wp_{33}(u)+\frac{1}{3}\lambda_2 Q_{2444}(u)+\left(\frac{1}{3}\lambda_1+\lambda_4\lambda_2-\frac{3}{4}\lambda_3^2\right)\wp_{33}(u)=0.\end{aligned} \tag{8.3}$$

At first order, we obtain four equations, for instance:

Corollary 8.5.

$$\begin{aligned}&-\wp_{11}(u)\wp_{144}(u)+\wp_{12}(u)\wp_{124}(u)-\frac{3}{4}\wp_{14}(u)\wp_{122}(u)+\frac{1}{3}\wp_{13}(u)(Q_{2444}(u))_1+\frac{1}{12}\wp_{14}(u)(Q_{3333}(u))_1\\&\quad+\frac{1}{6}\wp_{23}(u)(Q_{2333}(u))_1+\frac{1}{3}\wp_{33}(u)(Q_{1334}(u))_1-\frac{1}{3}\wp_{34}(u)(Q_{1333}(v))_1+\frac{1}{6}\lambda_3 Q_{2333}(u)\\&\quad-\frac{1}{2}\lambda_3\wp_{23}(u)\wp_{133}(u)-\wp_{111}(u)\wp_{44}(u)+\wp_{112}(u)\wp_{24}(u)-\frac{3}{4}\wp_{114}(u)\wp_{22}(u)\\&\quad+\frac{1}{3}\wp_{113}(u)Q_{2444}(u)+\frac{1}{12}\wp_{114}(u)Q_{3333}(u)+\frac{1}{6}\wp_{123}(u)Q_{2333}(u)+\frac{1}{3}\wp_{133}(u)Q_{1334}(u)\\&\quad-\frac{1}{3}\wp_{134}(u)Q_{1333}(u)-\frac{1}{12}(Q_{2222}(u))_1-\frac{1}{3}\lambda_4(Q_{1333}(u))_1+\frac{1}{6}\lambda_3(Q_{2333}(u))_1-\frac{1}{2}\lambda_3\wp_{123}(u)\wp_{33}(u)\\&\quad+\frac{1}{3}\lambda_2(Q_{2444}(u))_1+\left(\frac{1}{3}\lambda_1+\lambda_4\lambda_2-\frac{3}{4}\lambda_3^2\right)\wp_{133}(u)=0.\end{aligned}\tag{8.4}$$

At second order, we obtain several identities, in particular a "double-angle" sigma formula:

Corollary 8.6.

$$\begin{aligned}\frac{\sigma(2u)}{\sigma(u)^4}&=\frac{1}{6}\wp_{33}\wp_{122334}-\frac{7}{12}\wp_{1224}\wp_{33}^2-\frac{3}{4}\lambda_3\wp_{23}\wp_{2233}+\frac{1}{2}\wp_{222}^2-\frac{1}{24}\wp_{222222}\\&\quad+\frac{1}{6}\lambda_2\wp_{222444}+\frac{1}{2}\lambda_1\wp_{2233}-\frac{3}{8}\lambda_3^2\wp_{2233}+\frac{1}{6}\wp_{1223}\wp_{2444}+\frac{1}{6}\wp_{2233}\wp_{1334}\\&\quad+\frac{1}{24}\wp_{1224}\wp_{3333}+\frac{1}{24}\wp_{14}\wp_{223333}+\frac{1}{6}\wp_{13}\wp_{222444}-\wp_{23}\wp_{233}\wp_{223}\\&\quad+\frac{1}{12}\wp_{23}\wp_{222333}-\frac{1}{2}\wp_{2233}\wp_{23}^2+\frac{1}{3}\wp_{33}\wp_{2234}\wp_{13}+\frac{1}{12}\wp_{2223}\wp_{2333}-\frac{1}{2}\wp_{14}\wp_{233}^2\\&\quad-\lambda_2\wp_{44}\wp_{2224}-2\lambda_2\wp_{244}\wp_{224}-\lambda_2\wp_{2244}\wp_{24}+\frac{1}{2}\lambda_2\lambda_4\wp_{2233}\\&\quad-\wp_{1223}\wp_{44}\wp_{24}-\wp_{13}\wp_{44}\wp_{2244}-2\wp_{13}\wp_{2244}\wp_{24}-\wp_{23}\wp_{33}\wp_{2223}\\&\quad-\frac{7}{6}\wp_{33}\wp_{14}\wp_{2233}-\frac{2}{3}\wp_{33}\wp_{124}\wp_{233}-\frac{4}{3}\wp_{33}\wp_{234}\wp_{123}+\frac{1}{3}\wp_{34}\wp_{33}\wp_{1233}\\&\quad+2\wp_{34}\wp_{233}\wp_{123}+\frac{1}{3}\wp_{34}\wp_{2233}\wp_{13}+\lambda_4\wp_{33}\wp_{1233}+2\lambda_4\wp_{233}\wp_{123}\\&\quad+\lambda_4\wp_{2233}\wp_{13}-\lambda_3\wp_{233}\wp_{223}-\frac{3}{4}\lambda_3\wp_{2223}\wp_{33}-\frac{3}{8}\wp_{1224}\wp_{22}+\frac{1}{2}\wp_{1222}\wp_{24}\\&\quad-\frac{1}{2}\wp_{1122}\wp_{44}+\frac{1}{2}\wp_{12}\wp_{2224}-\frac{1}{2}\wp_{11}\wp_{2244}+\frac{1}{2}\wp_{2222}\wp_{22}\\&\quad+\frac{1}{12}\lambda_3\wp_{222333}-\frac{3}{8}\wp_{14}\wp_{2222}-\frac{1}{6}\lambda_4\wp_{122333}-\frac{1}{6}\wp_{34}\wp_{122333}.\end{aligned}$$

We expect that with better understanding of the PDEs satisfied by the $\wp_{ij}$, the right hand side of this formula could be simplified considerably.

9. Towards a three-term addition theorem

There must be a second main type of addition result satisfied by the σ-function of any cyclic trigonal curve:

Remark 9.1. The following function associated with (2.2),:

$$\frac{\sigma(u+v+w)\sigma(u+[\zeta]v+[\zeta^2]w)\sigma(u+[\zeta^2]v+[\zeta]w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3}$$

is Abelian in u, v and w, for it is a quotient of third-order theta functions in each argument.

It has triple poles where any of the arguments u, v or $w \in \Theta^{[3]}$. The left hand side thus belongs to

$$\Gamma(J \times J \times J, \mathcal{O}(3((\Theta^{[3]} \times J \times J) \cup (J \times \Theta^{[3]} \times J) \cup (J \times J \times \Theta^{[3]})))).$$

It must therefore possess an expansion of the form

$$\frac{\sigma(u+v+w)\sigma(u+[\zeta]v+[\zeta^2]w)\sigma(u+[\zeta^2]v+[\zeta]w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3} = \sum_{i=1}^{81}\sum_{j=1}^{81}\sum_{k=1}^{81} c_{ijk}U_i(u)V_j(v)W_k(w), \tag{9.1}$$

where the functions U_i, $V_j W_k$ are all basis functions for the space $\Gamma(J, \mathcal{O}(3\Theta^{[3]}))$.

Remark 9.2. This calculation may be reduced in complexity, by first finding an analogous, but less symmetric, expansion of the form

$$\frac{\sigma(u+v)\sigma(u+[\zeta]v)\sigma(u+[\zeta^2]v)}{\sigma(u)^3\sigma(v)^3} = \sum_{i=1}^{81}\sum_{j=1}^{81} d_{ij}U_i(u)V_j(v), \tag{9.2}$$

we note that the Sato weight of the left hand side of (9.2) is -24, so for each term in the sum, the Sato weights of the coefficient d_{ij} and the basis functions $U_i(u)$, $V_j(v)$ must sum to -24 also.

Remark 9.3. By applying

$$\frac{1}{3}\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j} + \frac{\partial}{\partial v_j} + \frac{\partial}{\partial w_j}\right)\log \tag{9.3}$$

to a formula like Remark 9.1, we would obtain algebraic addition formulae of another type, including a triple-angle formula. It would be interesting to compare such formulae with those of Remark 8.2.

Again, the difficulty of constructing the three-term formula explicitly should be considerably reduced when we note that every term in the sum must have total Sato weight -48, and that both sides are symmetric in u, v, and w, and invariant under $u \mapsto [\zeta]u$.

Such three-term addition formulae have been found explicitly for the cases of the equianharmonic elliptic curve and for the cyclic (3, 4) curve; this can be done wherever a basis of $\Gamma(J, \mathcal{O}(3\Theta^{[g-1]}))$ is known. As the dimension of this space is 3^g, such calculations for higher genus curves will rapidly become very unwieldy.

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